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On the Homotopy Index of Conley

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Isolating blocks and isolated invariant sets have become the focus for an extensive theory. One fundamental result, proven in various contexts by Conley and Easton [3], Wilson and Yorke [5], and Churchill [1], is that a perturbed isolating block remains in one form or another after the flow is perturbed. This result allows one to define a *continuation* of an isolated invariant set in a block to a nearby flow; namely, it is the invariant set for the nearby flow which is isolated by the corresponding block for the nearby flow. In other words, the set $\mathscr{S}$ of pairs consisting of a flow with one of its isolated invariant sets can be topologized by allowing a neighborhood of such a pair (f, S) to be formed by choosing a small neighborhood Φ of f in the space of flows, choosing a block for S, and letting the open set contain all pairs (f', S') with f' in Φ and S' isolated by a "perturbation" B' of B in the sense of Churchill's Perturbation Theorem for Blocks (Theorem 2). With this topology, $\mathscr{S}$ is a sheaf over the space of flows.

The index was introduced as the relative cohomology of a block modulo its exit set b^- [1], or as the homotopy type of a block with b^- identified to a point [4]. Both these definitions are independent of the particular block used. Also in [4] was developed a method of continuing any single element of the cohomology of the index of the invariant set as the flow is perturbed. This method entails constructing the set $\mathscr{H}$ (called $\mathscr{H}[i^-]$ in [4]) of triples (f, S, α) where (f, S) is in $\mathscr{S}$ and α is in the cohomology of the index of (f, S). In order to topologize $\mathscr{H}$, use must be made of the aforementioned Perturbation Theorem in which is provided a homotopy equivalence between the indices defined by a block and a perturbation of it. Thus to define an open set in $\mathscr{H}$, choose Φ and B as before and an element α of $H^*(B, b^-)$. Then a neighborhood of (f, S, α) is the set of (f', S', α') with (f', S') a continuation of (f, S) in $\mathscr{S}$, and α' the image of α under the map of cohomologies induced by the homotopy equivalence.

Under this topology, $\mathscr{H}$ becomes a sheaf over $\mathscr{S}$; in fact, $\mathscr{H}$ is locally a product over $\mathscr{S}$. This is stated in [4], but the proof was left incomplete. Rather than give the rest of the proof here—it is not difficult—we show how some of the results of this paper imply the local product property of $\mathscr{H}$.

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In this paper a conjecture of Conley is answered in the affirmative: given a flow and one of its isolated invariant sets, there is a neighborhood of the flow so that if the isolated invariant set is continued around any loop in the neighborhood, the resulting map of the index of the invariant set into itself is homotopic to the identity. In fact, on the space of flows with associated invariant sets, a *local system* is constructed over $\mathscr{S}$—to each homotopy class of paths from one isolated invariant set of a flow to another isolated invariant set of another flow, there is associated a unique homotopy class of homotopy equivalences from the corresponding indices.

The author is indebted to Charles Conley and to Sol Schwartzman for several very helpful conversations on this matter. Conley has also proven similar theorems in a different context and using different techniques in [2]. The author also thanks the referee for helpful suggestions.

We now state some facts and definitions about isolated invariant sets that we shall need later. Most of the proofs can be found in [4], among other places.

All flows mentioned in the paper are defined on a fixed compact metric space. The set of such flows, endowed with the compact-open topology, we denote by F. An *isolated invariant set* S for a flow f is an invariant (in both time directions) set which is maximal in some neighborhood of itself. Any proper closed subneighborhood containing S is an *isolating neighborhood*.

There are various kinds of *isolating blocks* constructed for various purposes, and each has its own technical definition. These definitions are not of use here and we refer the reader to [4] rather than cite them. Instead, we give a more intuitive definition suggested by Easton which better fits our needs. B is an *isolating block* for (f, S) if S is an isolated invariant set for f, and B is an isolating neighborhood for S with the property that the two maps associating to a point in B the time it takes for a point to leave B in the positive and negative time directions are continuous. This gives the boundary of B a certain structure: We let b^+ denote the set of points in the boundary which enter in the positive time direction and exit in negative time; b^- those which enter in the negative time direction and leave in positive time; and b^0 those points in the boundary whose orbits hit b^+ going backward and b^- going forward, and which stay in the boundary while doing this. The boundary of B is the union of these three closed sets. We use A^+ and A^- to denote the set of points in B which never leave B in positive and negative time respectively, and a^+ and a^- denote $b^+ \cap A^+$ and $b^- \cap A^-$ respectively.

Isolating blocks exist in arbitrarily small neighborhoods of the isolated invariant set, but no matter which of these blocks is used, the homotopy type of the pointed space B/b^- obtained from B by identifying b^- to a point is always the same. (Here and in the following, b^+ can replace b^-.) This follows from two facts. The first is that a new block $\bar{B}$ can be made from B in one of two ways with $\bar{B}/\bar{b}^-$ obviously keeping the same homotopy type as B/b^-: Let U be the complement of the closure of a neighborhood of a^+ in b^+, and for $C \subset D$ let

$P(C, D, f)$ denote the union of all positive orbit segments in D initiating from a point of C; i.e., $P(C, D; f) = \bigcup \{f(c, [0, t]) \mid c \in C, t \geq 0, \text{and} f(c, [0, t]) \subset D\}$. Then $\bar{B}$ is a *shave* of B if $\bar{B} = B - P(U, B; f)$. On the other hand, we say B^- is a *collar* of b^- if it is the union over b^- of orbit segments of the form $f(\{p\} \times [-r(p), 0]) \subset B$ where $p \in b^-$ and $r: b^- \to [0, \infty)$. B^- is an *open* collar if $r > 0$. Then $\bar{B}$ is a *squeeze* of B if $\bar{B} = B - \operatorname{int} B^-$ for some collar B^- of b^-. In the first case $(\bar{B}, \bar{b}^-) \subset (B, b^-)$ and the inclusion induces a homotopy equivalence of $\bar{B}/\bar{b}^-$ to B/b^-. In the second case, the inclusions $(\bar{B}, \bar{b}^-) \subset (B, B^-) \supset (B, b^-)$ all induce homotopy equivalences $\bar{B}/\bar{b} \to B/B^- \leftarrow B/b^-$. If $\bar{B}$ is a shave and a squeeze of B, we write $\bar{B} > B$; then there exists a homotopy equivalence from $\bar{B}/\bar{b}^-$ to B/b^-.

The second fact from which the invariance of the homotopy type of B/b^- follows is the Directed System Theorem.

THEOREM 1 (DST). *The set of $\{B/b^-\}$ for (f, S), together with the homotopy equivalences discussed above, is a directed system; that is, if B and $\bar{B}$ are two blocks for (f, S) then there is a third block $\tilde{B}$ such that $\tilde{B} > B, \bar{B}$. Furthermore, $\tilde{b}^- \subset b^- \cup \bar{b}^-$.*

Consequently, there is a homotopy equivalence defined between B/b^- and $\bar{B}/\bar{b}^-$ for any two blocks B and $\bar{B}$ for (f, S). We let $I(f, S)$ denote this directed system, which we call the *index* of (f, S).

If N is an isolating neighborhood for (f, S), then N is also an isolating neighborhood for every flow in a neighborhood $\Phi(N)$ of f in F. Then it is not hard to show that sets of the form $U(\Phi, N) \equiv \{(f', S') \mid f' \in \Phi,\ S'$ is an isolated invariant set for f', and N is an isolating neighborhood for $(f', S')\}$, where $\Phi \subset \Phi(N)$ is open, form a basis for a topology on the space $\mathscr{S}$ of all pairs (f, S). It is easily seen that the projection map $\pi: \mathscr{S} \to F$ when restricted to $U(\Phi, N)$ is a homeomorphism onto Φ; thus $\mathscr{S}$ is a sheaf over F.

We will be dealing with blocks for several different points in $\mathscr{S}$. We shall use the letter "B" to denote a block (with a bar or tilde) and we shall subscript with a point of $\mathscr{S}$; thus if $x = (f, S) \in \mathscr{S}$, then B_x denotes a block for x.

If $h: B_x/b_x^- \to B_y/b_y^-$, then its homotopy type is considered as a map $h: I(x) \to I(y)$. If $g: \bar{B}_x/\bar{b}_x^- \to \bar{B}_y/\bar{b}_y^-$, then as maps of the indices, $g = h$ if for all blocks $\tilde{B}_x > \bar{B}_x$, B_x and $\tilde{B}_y > B_y$, $\bar{B}_y$, the composition $\tilde{B}_x/\tilde{b}_x^- \leftrightarrow B_x/b_x^- \to^h B_y/b_y^- \leftrightarrow \tilde{B}_y/\tilde{b}_y^-$ has the same homotopy type as the composition $\tilde{B}_x/\tilde{b}_x \leftrightarrow \bar{B}_x/\bar{b}_x^- \to \bar{B}_y/\bar{b}_y^- \leftrightarrow \tilde{B}_y/\tilde{b}_y^-$. (The double arrows indicate homotopy equivalences in $I(x)$ or $I(y)$.) If $h: B_x/b_x^- \to B_y/b_y^-$ and $g: \bar{B}_y/\bar{b}_y^- \to B_z/b_z^-$, then the composition $g \circ h$ as a map of $I(x) \to I(z)$ is defined by choosing $\tilde{B}_y > B_y$, $\bar{B}_y$ and considering $B_x/b_x^- \to^h B_y/b_y^- \leftrightarrow \tilde{B}_y/\tilde{b}_y^- \leftrightarrow \bar{B}_y/\bar{b}_y^- \to^g B_z/b_z^-$ as a map of $I(x) \to I(z)$. The Corollary to the Triangle Lemma below implies that the choice of B does not affect the homotopy type of this last composition, and thus $g \circ h$ is independent of $\tilde{B}$.

TRIANGLE LEMMA. *Suppose X, Y, Z are topological spaces and $a\colon X \to Y$, $b\colon X \to Z$, and $c\colon Y \to Z$ are maps such that $b = ca$. Then if any two of a, b, c are homotopy equivalences, then so is the third, and any triangular diagram formed with any three of these maps or their inverses is commutative, at least up to homotopy.*

COROLLARY. *If the mappings in the diagram below are all homotopy equivalences and the diagram is commutative, then $c^{-1}b$ is homotopic to $e^{-1}d$.*

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "b"] \arrow[rr] \arrow[dr, "d"'] & & Y \arrow[ul, "c"'] \arrow[dl, "e"] \\
& W &
\end{tikzcd}$$

We now state Churchill's Perturbation Theorem for Blocks [1] which allows us to deal with isolated invariant sets of perturbations of a given flow. We state it here in a form more suitable to us, as developed in 6.6–6.10 of [4].

THEOREM 2 (CPTB). *Let $x = (f, S) \in \mathscr{S}$ and N be an isolating neighborhood for (f, S). Then there exists a block $B_x \subset N$ with open collar B_x^-, a neighborhood $U = U(B_x, B_x^-)$ of x in $\mathscr{S}$, and a collection of blocks $\{B_z \mid z \in U\}$ with the property that for all $y \in U$, we have $(B_y, b_y^-) \subset (B_x, B_x^-)$ and this inclusion induces a homotopy equivalence, denoted $j(B_y, B_x)$, from $I(y)$ to $I(x)$.*

Remark on notation. Whenever we use the notation $j(B_y, B_x)$ we will implicitly assume that x, y, B_x, B_x^- and B_y are as above. Similarly, use of the notation $V(B_x, B_x^-)$ will imply that x, B_x, and B_x^- are as in Lemma 1 below.

LEMMA 1. *Let x, B_x, B_x^- be as in* CPTB *as above. Then the following hold:*

(a) *There is a neighborhood $V = V(B_x, B_x^-)$ of x in $U(B_x, B_x^-)$ such that if $y \in V$ and B_y, $\bar{B}_y$ satisfy* CPTB, *then $j \equiv j(B_y, B_x) = j(\bar{B}_y, B_x) \equiv \bar{j}$ as maps of $I(y)$ into $I(x)$.*

(b) *Suppose x, $\bar{B}_x$, $\bar{B}_x^-$ are also as in* CPTB. *Then there exists an open set $W = W(B_x, B_x^-, \bar{B}_x, \bar{B}_x^-)$ in $V(B_x, B_x^-) \cap V(\bar{B}_x, \bar{B}_x^-)$ such that if $y \in W$, we have $j(B_y, B_x) = j(\bar{B}_y, \bar{B}_x)$ as maps of $I(y)$ into $I(x)$.*

(c) *Let $V_x \subset V(B_x, B_x^-)$ and $V_y \subset V(B_y, B_y^-)$, and suppose $z \in V_x \cap V_y$. Then as maps of $I(x)$ into $I(y)$, we have*

$$j(\bar{B}_z, B_y)\, j(B_z, B_x)^{-1} = j(\bar{B}_w, B_y)\, j(B_w, B_x)^{-1}$$

for all w in a closed and open (rel. $V_x \cap V_y$) subset W of $V_x \cap V_y$ which contains z.

Proof. (a) In order that $j = \bar{j}$ as maps of indices, then for $\tilde{B}_y > B_y$, $\bar{B}_y$, the following diagram must commute:

$$\begin{tikzcd}
 & B_y/B_y^- & B_y/b_y^- \arrow[l] \arrow[dr] & & \\
\tilde{B}_y/\tilde{b}_y^- \arrow[ur] \arrow[rrr] \arrow[dr] & & & B_x/B_x^- & B_x/b_x^- \arrow[l] \\
 & \bar{B}_y/\bar{B}_y^- & \bar{B}_y/\bar{b}_y^- \arrow[l] \arrow[ur] & &
\end{tikzcd}$$

Here, B_y^- and $\bar{B}_y^-$ are collars of b_y^- and $\bar{b}_y^-$ (resp.) which contain $\tilde{b}_y^-$; the block $\tilde{B}_y$ is provided by DST, so $\tilde{b}_y^- \subset b_y^- \cup \bar{b}_y^- \subset B_x^-$. Thus all maps are inclusion induced. However, the Triangle Lemma does not apply in this situation. In fact it is not very hard to construct an example for which the diagram is not commutative. The key to such a counterexample is that the backward f_y-orbits of $\tilde{b}_y^-$ in B_x might intersect A_x^+.

What is needed to apply the Triangle Lemma is a collar $\tilde{B}_x^- \supset B_x^-$ of b_x^- which not only contains $\tilde{b}_y^-$, but also its positive orbit segments in B_x. In this case we would have $B_y^- \equiv P(\tilde{b}_y^-, B_y\, ; f_y) \subset P(\tilde{b}_y^-, B_x\, ; f_y) \subset \tilde{B}_x^-$ and similarly $\bar{B}_y^- \equiv P(\tilde{b}_y^-, \bar{B}_y\, ; f) \subset \tilde{B}_x^-$. Then B_y^- and $\bar{B}_y^-$ would be collars of b_y^- and $\bar{b}_y^-$ (resp.), and $\tilde{B}_x^-$ could be included in the above diagram as follows:

$$\begin{tikzcd}
 & B_y/B_y^- \arrow[dr] & & B_y/b_y^- \arrow[ll] \arrow[dl] \arrow[dr] & & \\
\tilde{B}_y/\tilde{b}_y^- \arrow[ur] \arrow[rr] \arrow[dr] & & B_x/\tilde{B}_x^- & & B_x/B_x^- \arrow[ll] & B_x/b_x^-. \arrow[l] \\
 & \bar{B}_y/\bar{B}_y^- \arrow[ur] & & \bar{B}_y/\bar{b}_y^- \arrow[ll] \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Now the Triangle Lemma and its corollary could be applied with the result that the map across the top would be homotopic to the one across the bottom, and thus $j = \bar{j}$ as maps of the indices.

We claim that such a $\tilde{B}_x^-$ exists if we let Φ be a neighborhood of f_x in F such that $f' \in \Phi$ implies that $P(B_x^-, B_x\, ; f') \subset B_x \backslash A_x^+$, and define $V = V(B_x, B_x^-) \equiv \{(f_y, S_y) \in U(B_x, B_x^-) \mid f_y \in \Phi\}$. In this case B_y^- and $\bar{B}_y^-$ as defined above are subsets of $B_x \backslash A_x^+$. The existence of $\tilde{B}_x^-$ follows from the fact that there is a collar of b_x^- containing *any* closed subset of $B_x \backslash A_x^+$. Thus (a) is proved.

(b) Select a block $\tilde{B}_x > B_x$, $\bar{B}_x$ and enlarge the collars B_x^- and $\bar{B}_x^-$ so that they each include a collar $\tilde{B}_x$ of $\tilde{b}_x^-$. Now if $y \in V(\tilde{B}_x, \tilde{B}_x^-)$, part (a) allows us to replace B_y and $\bar{B}_y$ with a block $(\tilde{B}_y, \tilde{b}_y^-) \subset (\tilde{B}_x, \tilde{B}_x^-)$. The following inclusion induced commutative (up to homotopy) diagram summarizes our situation:

$$\begin{tikzcd}
& B_x/B_x^- \arrow[dl] & \\
\tilde{B}_x/\tilde{B}_x^- \arrow[dr] & & \tilde{B}_y/\tilde{b}_y^- \arrow[ll] \arrow[ul] \arrow[dl] \\
& \bar{B}_x/\bar{B}_x^- &
\end{tikzcd}$$

Since the maps across the top and across the bottom are each homotopic to the one across the middle, part (b) follows.

(c) We must show that the following diagram of homotopy equivalences commutes, at least up to homotopy

$$\begin{tikzcd}
& B_z/B_z^- \arrow[dl] \arrow[r, leftrightarrow] & \bar{B}_z/\bar{B}_z^- \arrow[dr] & \\
B_x/B_x^- & & & B_y/B_y^-. \\
& B_w/b_w^- \arrow[ul] \arrow[uu] \arrow[r, leftrightarrow] & \bar{B}_w/\bar{b}_w^- \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Here the double arrows are maps within the directed system of blocks for z or w. Now part (b) implies the inside square commutes for $w \in W(B_z, B_z^-, \bar{B}_z, \bar{B}_z^-)$ where $B_z^- \subset B_x^-$ and $\bar{B}_z^- \subset B_y^-$. It follows that the set W of $w \in V_x \cap V_y$ for which the equation in (c) holds is open. Furthermore, since z was chosen arbitrarily in $V_x \cap V_y$, we have also shown that the complement of W in $V_x \cap V_y$ is open. This completes the proof of Lemma 1.

Definition. Suppose $x_i \in V_i \subset V(B_{x_i}, B_{x_i}^-)$ for $i = 0, 1$, and $y \in V_0 \cap V_1$. Then Lemma 1(a) implies that for all $z \in V_1$, the homotopy class of the composition

$$c = j(B_z, B_{x_1})^{-1} j(\bar{B}_y, B_{x_1}) j(B_y, B_{x_0})^{-1}$$

from $I(x_0)$ to $I(x_1)$ is independent of the choices of B_y and $\bar{B}_y$. Furthermore Lemma 1(c) implies the homotopy class of c is the same for all y in a closed and open (rel. $V_0 \cap V_1$) set W of $V_0 \cap V_1$. We say (V_0, V_1, L) is a ***link*** at x_0

(to x_1) if $V_0 \cap V_1 = W$, and for each $z \in V_0 \cap V_1$, $Lz: I(x_0) \to I(z)$ is defined as

$$Lz \equiv \begin{cases} j(B_z, B_{x_0})^{-1} & \text{if } z \in V_0 \\ c & \text{if } z \in V_1 \backslash V_0 \end{cases}$$

LEMMA 2. *If (V_0, V_1, L) is a link at x_0 and $z \in V_0 \cup V_1$, the* localization at z *of L is inclusion induced in the sense that z has a neighborhood W_z such that if $w \in W_z$, then*

$$Lz\,Lw^{-1} = j(B_w, B_z).$$

Proof. If $z \in V_0$, then if $w \in V_0 \cap V(B_z, B_z^-)$, where $(B_z, B_z^-) \subset (B_{x_0}, B_{x_0}^-)$, we have $Lz\,Lw^{-1} = j(B_z, B_{x_0})^{-1} j(B_w, B_{x_0})$. But since both maps of this latter composition are inclusion induced, it follows that the composition has the homotopy type of $j(B_w, B_z)$.

If $z \in \operatorname{int}(V_1 \backslash V_0)$ and $w \in \operatorname{int}(V_1 \backslash V_0) \cap V(B_z, B_z^-)$ for some $(B_z, B_z^-) \subset (B_{x_1}, B_{x_1}-)$, then

$$Lz\,Lw^{-1} = j(B_z, B_{x_1})^{-1} j(B_w, B_{x_1}) = j(B_w, B_z).$$

Finally, suppose z is in the boundary (rel. $V_1 \cup V_0$) of $V_1 \backslash V_0$. Since V_0 is open, then $z \in V_1 \backslash V_0$. Choose $(B_z, B_z^-) \subset (B_{x_1}, B_{x_1}^-)$; we have already seen that if $w \in V(B_z, B_z^-) \cap (V_1 \backslash V_0)$ then $Lz\,Lw^{-1} = j(B_w, B_z)$. We wish to show that this also holds if $w \in V(B_z, B_z^-) \cap (V_1 \cap V_0)$. But for such w

$$Lz\,Lw^{-1} = j(B_z, B_{x_1})^{-1} j(\bar{B}_y, B_{x_1})\, j(B_y, B_{x_0})^{-1} j(B_w, B_{x_0}).$$

Since (V_0, V_1, L) is a link at x and $w \in V_0 \cap V_1$, we can replace y, B_y and $\bar{B}_y$ in the above composition with w, B_w, and $\bar{B}_w$, where $(\bar{B}_w, \bar{b}_w^-) \subset (B_z, B_z^-)$. Thus

$$Lz\,Lw^{-1} = j(B_z, B_{x_1})^{-1} j(\bar{\bar{B}}_w, B_{x_1}) = j(\bar{B}_w, B_z).$$

THEOREM 3. *Suppose (U_0, U_1, K) and (V_0, V_1, L) are links at x_0. Then the set $\{z \mid Lz = Kz\}$ is open and closed in $(U_0 \cup U_1) \cap (V_0 \cup V_1)$. In particular, if both $U_0 \cup U_1$ and $V_0 \cup V_1$ contain an arc at x_0, then the link maps are equal at all points of the arc.*

Proof. That the set and its complement are both open follows from Lemma 2 and Lemma 1(b).

DEFINITION. (V_0, V_1, L) is a link at x_0 to x_1 *along the path* $\alpha: [t_0, t_1] \to \mathscr{S}$ from x_0 to x_1 if not only is it a link at x_0 to x_1 but also there exists t with $t_0 \leqslant t \leqslant t_1$ such that $V_0 \supset \alpha[t_0, t]$ and $V_1 \supset \alpha[t, t_1]$.

DEFINITION. A *chain* along a path $\alpha: [0, 1] \to \mathscr{S}$ is a collection $\{(s_i, U_i)\}_{i=0}^{m}$

such that $0 = s_0 \leqslant \cdots \leqslant s_m = 1$ and for each $i = 0, 1,..., m - 1$, (U_i, U_{i+1}, L_i) is a link at $x_i = \alpha(s_i)$ to $x_{i+1} = \alpha(s_{i+1})$ along $\alpha \mid [s_i, s_{i+1}]$. The *chain map* C along α associated to the chain is defined for each t in $[0, 1]$ by the following: if $t \in [s_i, s_{i+1}]$, then

$$C_t \equiv L_i\alpha(t) \circ L_{i-1}x_i \circ \cdots \circ L_1x_2 \circ L_0x_1 \colon I(x_0) \to I(\alpha(t)).$$

Note that the two definitions of $C_{s_{i+1}}$ amount to the same thing.

THEOREM 4. *Let x_0, x_1 be points in $\mathscr{S}$ and $\alpha\colon [0, 1] \to \mathscr{S}$ be a path connecting them. There is one and only one chain map $C[\alpha]$ along α. Furthermore, if α is homotopic to β with corresponding chain maps C and D, then $C_1 = D_1$.*

Proof. We first show uniqueness: Let $\{(s_i, U_i)\}_{i=0}^m$ and $\{(t_j, V_j)\}_{j=1}^n$ be chains along α, with associated link maps $\{K_j\}_{j=0}^{m-1}$ and $\{L_j\}_{j=0}^{n-1}$ and associated chain maps C_t and D_t respectively. Let $T \equiv \sup\{t \mid C_s = D_s \text{ for all } s \in [0, t]\}$. It follows from Theorem 2 that $T > 0$. In fact, if $s_i < T \leqslant s_{i+1}$ and $t_j < T \leqslant t_{j+1}$ with $s_i < t_j$, then for $s_i \leqslant t \leqslant s_{i+1}$, $\bar{C}_t \equiv C_tC_{s_i}^{-1} = K_i\alpha(t)$ and for $s_{i+1} \leqslant t$ with $\alpha[s_{i+1}, t] \subset U_{i+1}$, $\bar{C}_t \equiv C_tC_{s_i}^{-1} = K_{i+1}\alpha(t)\, K_i\alpha(s_i) = K_i\alpha(t)$. Similarly, for $t_j < t \leqslant t_{j+1}$, or $t_{j+1} \leqslant t$ such that $\alpha[t_{j+1}, t] \subset V_{j+1}$, we have $\bar{D}_t \equiv D_tD_{t_j}^{-1} = L_j\alpha(t)$. Thus Lemmas 2 and 1 now imply that the localizations of $\bar{C}_t$ and $\bar{D}_t$ at $\alpha(T)$ agree on a subarc of α with interior point $\alpha(T)$, i.e., for t near T, $\bar{C}_T\bar{C}_t^{-1} = \bar{D}_T\bar{D}_t^{-1}\colon I(\alpha(t)) \to I(\alpha(T))$. Writing this out, we see that $C_TC_t^{-1} = D_TD_t^{-1}$ for t near T. But for $t < T$ we have $C_t = D_t$, so it follows that $C_T = D_T$ and in fact that $C_t = D_t$ for all t near T. It now follows that $T = 1$ and that $C_t = D_t$ for all t in $[0, 1]$.

Now suppose α, $\{(s_i, U_i)\}_{i=0}^m$, $\{K_i\}_{i=0}^{m-1}$, C_t are as before and β is another path from $\alpha(0)$ to $\alpha(1)$ such that for each i there is an $\bar{s}_i$ such that $s_i < \bar{s}_i < s_{i+1}$ and $\beta[s_i, \bar{s}_i] \subset U_i$ and $\beta[\bar{s}_i, s_{i+1}] \subset U_{i+1}$. For $t \in [s_i, \bar{s}_{i+1}]$, define D_t as $K_i\beta(t) \circ K_{i-1}\alpha(s_i) \circ \cdots \circ K_0\alpha(s_1)$ (which is the same as $K_{i+1}\beta(t) \circ K_i\alpha(s_i) \circ \cdots \circ K_0\alpha(s_1)$ if $t \in [s_{i+1}, \bar{s}_{i+1}]$). Then for $t \in (s_i, \bar{s}_{i+1})$, define $\bar{D}_t \equiv D_tD_{s_i}^{-1} = K_i\beta(t)$. Then the localization of $\bar{D}_t$ is inclusion induced. It now follows as in the proof of uniqueness that D_t must be the same as the unique chain map along β. (We prove existence in a moment.) Since $D_1 = C_1$, we have shown that the chain map along α from $\alpha(0)$ to $\alpha(1)$ is the same as the chain map along β between the same two points as long as β is close enough to α (in the compact open topology). It follows that if α is homotopic to β (even if β is not close to α), then $C_1 = D_1$.

There remains only the chore of proving the existence of a chain along a given path α. Let $T = \sup\{t \mid \text{there is a chain along } \alpha[0, t]\}$. Now let $y = \alpha(T)$ and $V_y = V(B_y, B_y^-)$. Then V_y contains an open connected segment of α which contains y. Let $x = \alpha(t)$ for $t < T$ be some point of this segment, and let $\{(t_i, V_i)\}_{i=0}^n$ be a chain along α from $\alpha(0)$ to x. Now let V_{n+1} be the open and closed (rel. $V_n \cap V_y$) subset of $V_n \cap V_y$ provided by Lemma 1(c) (with

$V_x = V_n$ and $z = x$). If $t_{n+1} \equiv t_n$, then it is a trivial matter to check that $\{(t_i, V_i)\}_{i=0}^{n+1}$ is also a chain, and therefore, defining $t_{n+2} = T$ and $V_{n+2} = V_y$, we get that $\{(t_i, V_i)\}_{i=0}^{n+2}$ is a chain from $\alpha(0)$ to $\alpha(T)$.

If $T \neq 1$, this chain can be extended to a chain along $\alpha[0, T + \epsilon]$ for some $\epsilon > 0$ by letting $t_{n+3} > T$ such that $\alpha(t_{n+3}) \in V_{n+3} \subset V_{n+2}$. Thus $T = 1$ and the Theorem follows.

COROLLARY. *If $x \in \mathscr{S}$ and α is a loop in $V = V(B_x, B_x^-)$ at x, then $C[\alpha]$ is the identity on $I(x)$, whether or not α is contractible to a point in V.*

Proof. Let $t_0 = 0$, $t_1 = 1$, $V_0 = V_1 = V$. A chain along α is $(\{t_i\}, \{V_i\})_{i=0,1}$ and the associated chain map along α is the identity at $t = 1$.

Finally, we show how the results of this paper imply the following:

COROLLARY. *The sheaf $\mathscr{H}$ is locally a product over $\mathscr{S}$.*

Proof. Let $x = (f, S) \in \mathscr{S}$, $V \subset V(B, B^+)$, and let L be the link map associated to the link formed from $V_0 = V_1 = V$. Then for each $x' \in V$, $(Lx')^*$: $H^*(I(x)) \to H^*(I(x'))$. Define the one to one, onto map m: $V \times H^*(I(x)) \to \mathscr{H} \mid V$ by $m(x', \alpha) = (x', (Lx')^* \alpha)$. (The discrete topology is used for $H^*(I(x))$ in the product.) By insisting that all such maps are open, one determines the topology on $\mathscr{H}$ which was defined in [4].

Now suppose $\mathscr{U}$ is an open set in $\mathscr{H} \mid V$ and that $(x_1, \beta_1) = m(x, \beta)$ is an arbitrary point of $\mathscr{U}$. Then there is a point (x_0, β_0) in $\mathscr{H} \mid V$, a map m_0 and an open $V_0 \subset V$ of x_1 such that $\mathscr{U} \supset m_0(V_0 \times \{\beta_0\})$. In particular, $m_0(x_1, \beta_0) = m(x_1, \beta)$. If $V_1 \subset V_0$ is a neighborhood of x_1 and m_1: $V_1 \times H^*(I(x_1)) \to \mathscr{H} \mid V_1$ is any other map defined at x_1 as m was defined at x above, then Theorem 3 implies that there is a neighborhood $W \subset V_1$ of x_1 such that $m_0(w, \beta_0) = m_1(w, \beta_1)$ whenever $w \in W$. Now Theorem 3 also implies that there is a neighborhood W_1 of x_1 in W such that if $w \in W_1$, then $m(w, \beta) = m_1(w, \beta_1)$. It follows that $m(W_1 \times \{\beta\}) \subset U$, and hence that m is continuous, and a homeomorphism.

REFERENCES

1. R. CHURCHILL, Isolated invariant sets in compact metric spaces, *J. Differential Equations* **12** (1972), 330–352.
2. C. CONLEY, Isolated invariant sets and the Morse index, *NSF Regional Conference Series in Math.* **38** (1978).
3. C. CONLEY AND R. EASTON, Isolated invariant sets and isolating blocks, *Trans. Amer. Math. Soc.* **158** (1971), 35–61.
4. J. T. MONTGOMERY, Cohomology of isolated invariant sets under perturbation, *J. Differential Equations* **13** (1973), 257–299.
5. F. W. WILSON AND J. YORKE, Lyapunov functions and isolating blocks, *J. Differential Equations* **13** (1973), 106–123.